Chapter 2

Sudoku

2.1 Sudoku Rules

Regular Sudoku is a single-player puzzle consisting of a square 9×9 grid usually containing digits ranging from 1 through 9. A significant number of grid entries are empty. We call each of the digits that are given a **clue**. See Figures 2.1(a)-2.2(b) for a few example Sudoku puzzles. The large square is subdivided into nine 3×3 sub-squares. The goal is to fill the digits 1 through 9 into the open grid positions according to the following three rules:

- Each **row** must contain each of the digits 1 through 9 exactly once. (Notice that this means that each of those digits must occur somewhere along the row, and none of them can be duplicated.)
- Each **column** must contain each of the digits 1 through 9 exactly once.
- Each of the small 3×3 squares must contain each of the digits 1 through 9 exactly once.

Regular Sudoku puzzles are designed to have one unique solution, i.e. there is only one possible correct answer for each space in the grid. If you find yourself guessing between a variety of possible answers, you may guess incorrectly. Make sure you eliminate all but a single answer before filling in a final solution!

Sudoku puzzles are widely available in newspapers, books, and online. There is even an iPhone App.

Before we go any further, take a stab a solving a few puzzles. Figures 2.1-2.2 show four Sudoku puzzles of varying levels of difficulty: Figures 2.1(a)-2.1(b) are considered entry level, Figure 2.2(b) more challenging, with Figure 2.2(a) being somewhere in between.

1. Pick two of the puzzles in Figures 2.1-2.2 that are somewhat challenging for your level of experience with Sudoku. Solve these two puzzles. (If you are ready for an even more challenging Sudoku puzzles, you may replace these by more challenging ones. Clearly note your source.) As you solve the puzzle, keep an eye on the processes and the strategies you use.

2. **Classroom Discussion:** Summarize the processes and strategies you use to solve these puzzles and explain why they are helpful to you in solving the Sudoku puzzle.

3	2	4			9		7	
		7			2			8
8		6		5		1	2	
9	3		5	1				
			8		4			
				9	3		5	1
	4	3		7		8		9
6			9			4		
	8		6			5	3	7

(a) Entry level.

5					2		4	1
	4	1		8		3	6	
	6	3	9				8	
	5					6		
8		6		7		1		3
		9					7	
	3				8	2	9	
	9	5		2		8	1	
7	8		1					6

(b) Entry level.

Figure 2.1: Two Sudoku puzzles.

	3		5			7		
		6				9		
	4		9	2	7	1	6	
4		5		9	6			
			2		3			
			7	4		6		1
	5	2	8	7	9		1	
		4				5		
		3			4		9	

(a) Medium.

6			7			9		
								4
	2	9	1					6
		4	5		7			
9		5	4		6	2		3
			9		8	4		
2					1	3	9	
5								
		7			3			1

(b) More challenging.

Figure 2.2: Two Sudoku puzzles.

2.2 History

2.2.1 Latin Squares

Sudoku puzzles are an example of what the famous mathematician **Leonhard Euler** (Swiss mathematician and physicist; 1707 - 1783) called ***Latin Squares***: these are square tables filled with digits, letters, or symbols so that each of the entries occurs only once in each row or column. (Notice that Sudoku puzzles have the added requirement that each of the little 3×3 contain the digits 1–9 only once.) Figure 2.3 shows a beautiful example of a Latin Square which does not use digits or letters: in this stained glass window, each of the seven colors occurs only once in each row and in each column.

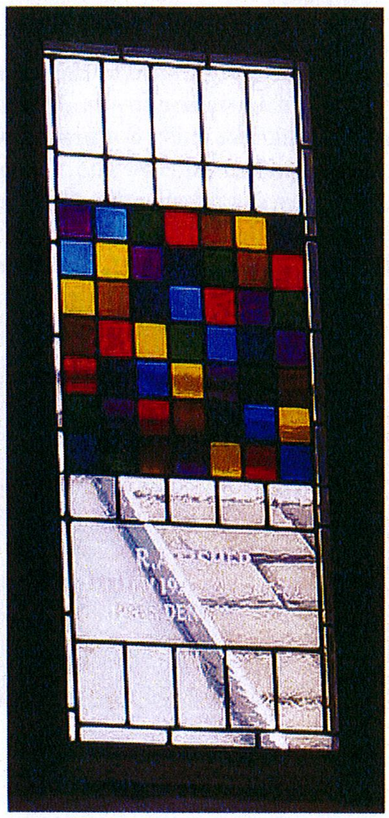

Figure 2.3: Displaying a Latin square, this stained glass window honors Ronald Fisher, whose "Design of Experiments" discussed Latin squares. Fisher's student, A. W. F. Edwards, designed this window for Caius College, Cambridge.

3. Working on your own, turn the color Latin Square displayed in the stained-glass window in Figure 2.3 into a Latin Square that uses numbers. Describe your process in detail. (If this printed document is not clear enough to make out the color detail, you will find a color image of the window at `http://en.wikipedia.org/wiki/File:Fisher-stainedglass-gonville-caius.jpg`.)

4. Now, find a different set of numbers that also fit this particular stained glass window. How do your results compare? Explain your observations.

5. Compare your results with those of other students. In what ways are the different Latin Square that were created the same? In what ways are they different? Explain.

6. How many different ways do you think there are to create a number version of the color Latin Square from Figure 2.3 (say, using the numbers 1 – 7)? Document your process of thinking about this question. Carefully explain your answers and ideas. Clearly explain why your reasoning and results makes sense.

Latin Squares were know long before Euler gave them this name, and in cultures far from the Latin world. You can find examples of Latin squares in Arabic literature over 700 years old. An ancient Chinese legend goes something like this: Some three thousand years ago, a great flood happened in China. In order to calm the vexed river god, the people made an offering to the river Lo, but he could not be appeased (Figure 2.4 shows a Ming dynasty scroll depicting the nymph of the Lo river). Each time they made an offering, a turtle would appear from the river. One day a boy noticed marks on the back of the turtle that seemed to represent the numbers 1 to 9. The numbers were arranged in such a way that each line added up to 15, as in Table 2.2.1. Hence the people understood that their offering was not the right amount.

Figure 2.4: Nymph of the Lo River, an ink drawing on a handscroll, Ming dynasty, 16th century. Freer Gallery of Art

4	9	2
3	5	7
8	1	6

Table 2.1: The Lo Shu magic square.

7. In the Lo Shu square, describe all the number patterns and relationships you can find.

Figure 2.5 shows one of the best-known depictions of a Latin Square in Western art. It is included in a woodcut called "Melencolia I" by the German artist **Albrecht Dürer** (painter, printmaker and theorist; 1471 - 1528), conventionally regarded as the greatest artist of the Northern Renaissance. Figure 2.6 shows "Melencolia" in its entirety. Dürer spent significant time in Italy and remained in communication with most of the major artists of the time, including **Raphael** (Italian painter; 1483 - 1520) and the proverbial Renaissance man **Leonardo da Vinci** (sculptor, painter, scientist; 1452 - 1519).

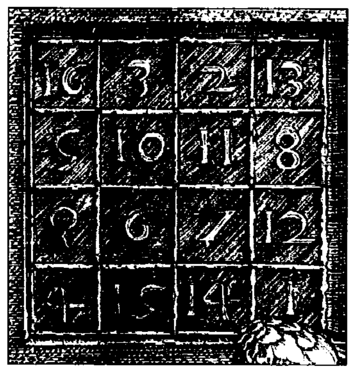

Figure 2.5: Latin Square from Dürer's woodcut *Melencolia.*

Figure 2.6: Albrecht Dürer created his famous woodcut *Melencolia* in 1514, showing a brooding figure which some believe is the prototype of Rodin's *The Thinker* and other renderings of people deep in thought. Notice the Latin Square to the right above the angel's head.

8. Transcribe the Latin Square from Dürer's woodcut Figure 2.5. Describe in detail as many patterns as you can find.

9. **Writing Assignment:** Research the history of Sudoku puzzles and write about your findings. Focus on two particular aspects that you find especially intriguing.

2.2.2 Sudoku

Source: Wikipedia,

- 18th century: Latin squares (Euler, Swiss)
- Late 1970's: Dell Magazines (US, "Number Place"), 1979.
- mid 1980 (1984): Nikoli (Japan, "Su: number", "Doku: single place"), 30 clues arranged symmetrically, Japan craze: 'Suuji wa dokushin ni kagiru' roughly translating to mean the numbers must be unmarried or single.
- Wayne Gould (NZ), Retired Hong Kong judge: computer program, "Su Doku"
- Nov 12, 2004: The Times (UK), "Su Doku"
- 2004: Wayne Gould, Conway Daily Sun (NH)
- 2005: worldwide craze
- 2008: "After 105 witnesses and three months of evidence, a drug trial costing $1 million was aborted yesterday when it emerged that jurors had been playing Sudoku since the trial's second week." [1]

[1] Knox, Malcolm (2008-06-11). "The game's up: jurors playing Sudoku abort trial". The Sydney Morning Herald.

2.3 Removing Clues

You may have noticed that the Sudoku puzzles in Figures 2.1-2.2 have different numbers of clues. How does this compare with the difficulty of these puzzles? One way we can try to make a puzzle more difficult (or at least require more work) is to remove some of the clues. (This is not the only way; see Section 2.4 for more information on creating puzzles.)

7			3		9	4	6	
4		1			2		8	9
6	3			8		2		
		4					1	
2	7						9	4
	1					7		
		7		2			3	6
1	6		5			9		7
	9	2	6		3			8

Figure 2.7: Remove one digit, making sure it still has a unique solution.

10. Classroom Discussion: In Figure 2.7, you can remove a "7" from the puzzle so that it still has a unique solution. Why can you be sure that the puzzle is still unique? Are there any other digits you can safely remove?

11. For the Sudoku puzzle in Figure 2.1(a), find one digit you could safely remove from the puzzle. (Make sure that it still has one unique solution!)

12. For the Sudoku puzzle in Figure 2.1(b), find one digit you could safely remove from the puzzle.

13. For the Sudoku puzzle in Figure 2.2(a), find one digit you could safely remove from the puzzle.

2.4 Creating Your Own Puzzles

4			
		1	
	2	3	
			4

(a) Joe's Sudoku puzzle.

		1	3
	2		
4			

(b) Karen's Sudoku puzzle.

Figure 2.8: Two 4×4 Sudoku puzzles.

14. Joe invented the Sudoku puzzle in Figure 2.8(a). What do you notice as you solve the puzzle?

15. Karen invented the Sudoku puzzle in Figure 2.8(b). What do you notice as you solve the puzzle?

16. It seems that creating Sudoku puzzles is not that easy. How do you think Joe and Karen should have gone about creating such puzzles?

17. Create four different puzzles yourself, then exchange them with a partner. Solve the puzzles you receive. Does each have a solution? Is it unique? (Figure 2.9 on page 45 has empty templates for 4×4 Sudoku puzzles.)

18. **Classroom Discussion:** From among the puzzles you created, pick your favorite one and share it with the class. Comparing all the puzzles, your creation strategies and the solutions, what patterns, similarities, and differences do you notice?

19. INDEPENDENT INVESTIGATION: It is an open problem to know what is the smallest number of clues needed for a 9×9 Sudoku puzzle[2] (with a unique solution). Investigate what is the smallest number of clues needed for a 4×4 puzzle (with a unique solution)?

20. INDEPENDENT INVESTIGATION: What is the **largest** number of clues you can have in a 4×4 puzzle that still has **no unique** solution?

For 9×9, the largest possible number of clues you can have without a unique solution is to have all but four squares filled with clues (81-4 = 77). Could you beat that for the 4×4 (i.e. have fewer than 4 open squares and still have no unique solution)? How about other size puzzles?

2.5 Sudoku Competition

21. Writing Assignment: What can you find out about Sudoku competitions around the world? What material did you come across in your research that you find most interesting. Write about your research.

2.6 3D variants

22. Writing Assignment: How could we create a three-dimensional version of Sudoku? Write about your investigations (you can do research or create your own) and give examples.

2.7 Appendix: 4×4 Worksheets

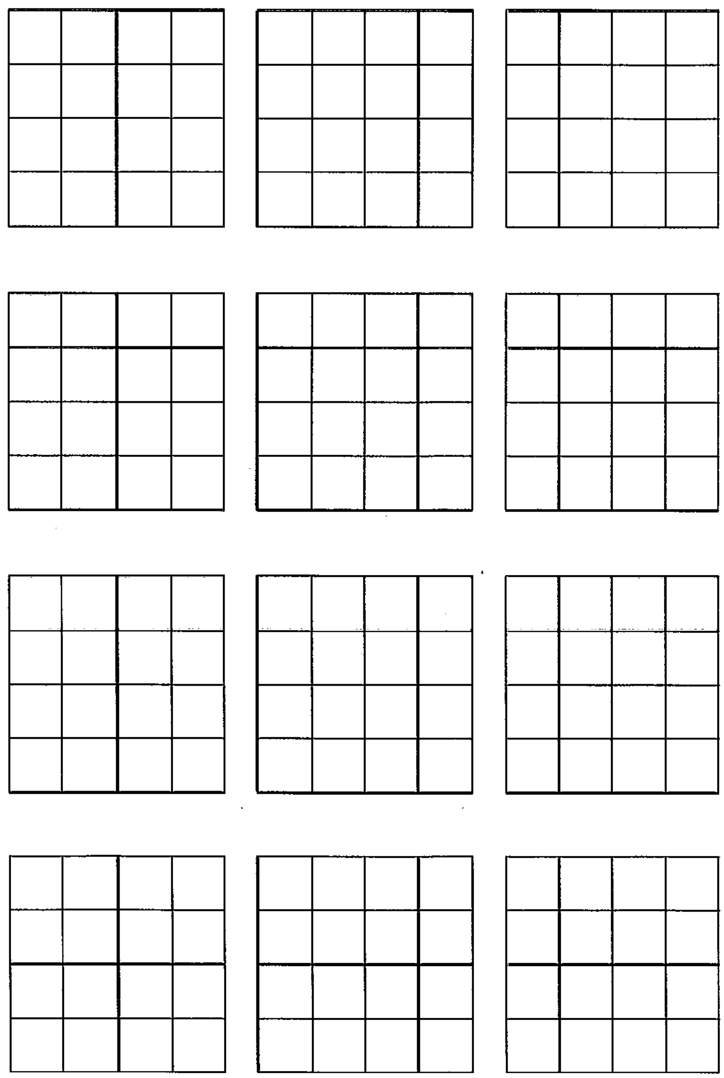

Figure 2.9: Templates for 4×4 Sudoku puzzles.