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Mathematical Magic for Muggles

Here are several easy-to-perform feats that suggest supernatural powers such as telepathy, "seeing fingers," predicting the future, photographic memory, etc. Each trick uses simple mathematical ideas that allow information to flow effortlessly and sneakily, among them

- parity and other invariants
- symmetry
- probability

One can approach these activities in many ways. At first, you may want to figure out HOW to do a trick. Then, you want to know WHY it works. Finally, you should strive to understand REALLY WHY it works: is there a simple theme or principle behind your possibly complex explanation? Look for simple and general guiding principles.

Several of these tricks were researched, perfected, and classroom-tested in 2012 at the San Francisco Math Circle by SFSU grad students Jessica Delgado and Kelly Walker. I am indebted to them. In turn, they (and I) are also indebted to the recent *Magical Mathematics*, by Persi Diaconis and Ron Graham (Princeton University Press, 2012).

Mostly about Coding and Communication

1 *Warm-up: Fingers That Can See*. The Magician deals cards on a table (not in a pile), placing them face up or face down on the command of the Participant, and stops dealing when the Participant says so.

Then the Magician is blindfolded. The Magician proceeds to put the cards into two piles, using his magical seeing fingers, so that, miraculously, each pile has exactly the same number of face-up cards!

2 *Zvonkin's Magic Table*. This trick is adapted from A. Zvonkin's book *Math From 3 to 7*, which I helped to translate and edit. Zvonkin ran a math circle for small kids in Moscow and entertained them by having them cover any four consecutive numbers in the table below (vertical or horizontal), and then he would instantly determine the sum! Was it a feat of memory? Telepathy?

5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0
7	1	7	2	3	7	1	7	2	3	7	1
2	7	6	1	4	2	7	6	1	4	2	7
5	6	1	6	2	5	6	1	6	2	5	6
5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0
7	1	7	2	3	7	1	7	2	3	7	1
2	7	6	1	4	2	7	6	1	4	2	7
5	6	1	6	2	5	6	1	6	2	5	6
5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0

3 *The Kruskal Count*. This telepathy trick can be done with cards or numbers. With cards, the Magician deals out an entire deck face up on a table, and asks the participant to mentally pick one of the first dozen or so cards and then use that card to tell him or her where to go next. If the card is an Ace, move one spot to the next card. If it's 2 through 9, go that many places. If it's a face card, move the number of letter of the card (i.e., Jack or King means move four, Queen means move five). Keep doing this until you can go no further. For example, if you start with the Jack of Hearts, you then move 4 cards down and perhaps that is an Ace of clubs. Then you move to the next card, the 7 of spades, and move 7 down, etc.

When the participant gets to the final card (the one where you cannot go further, because you'd go past the last card in the deck), he or she thinks hard about it. And the Magician manages to deduce the card.

The trick can also use a random list of numbers, or a semi-random one, such as the digits of π below.

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4
6	2	6	4	3	3	8	3	2	7	9	5	0	2	8	8	4	1	9	7
1	6	9	3	9	9	3	7	5	1	0	5	8	2	0	9	7	4	9	4
4	5	9	2	3	0	7	8	1	6	4	0	6	2	8	6	2	0	8	9
9	8	6	2	8	0	3	4	8	2	5	3	4	2	1	1	7	0	6	7

With a number table, the rule is simpler: Pick any starting point in the row, and move that many places, unless you hit 0, in which case you move one place. For example, if you start with the second digit (1), you move one place, to 4, then 4 more places, to 2, then 2 places, to 5, etc. Once again, the Participant mentally chooses a starting point, concentrates on the ending number, and the Magician magically guesses it!

4 *Hummer Shuffle Tricks*. The three tricks below all employ the "Hummer Shuffle," which consists of picking up the first two cards of a deck, turning the two cards over, and replacing them on the top of the deck (i.e., card #1 becomes card #2 and card #2 becomes card #1, and both get turned over), followed by cutting the deck (you take the top n cards, where n is up to you, and lift them off the deck, then place them at the bottom, without turning the n cards over, so that now the top card is the previous $(n+1)$st and the bottom card is the previous nth card, etc. After doing a bunch of Hummer Shuffles, the cards in a deck are hopelessly messed up, since not only is the order permuted, but some of the cards will be face up and some will be face down. However, this shuffle is surprisingly orderly, as you will see.

(a) *Baby Hummer*. This trick only uses four cards. The Participant takes four cards, all facing the same way, and sneaks a peek at the bottom card. Then the Participant does the following:

1. Take the top card and place it on the bottom
2. Turn the current top card face up
3. Perform several Hummer Shuffles
4. Turn over the top card and put it on bottom
5. Put the current top card on the bottom without turning it over
6. Turn the top card over and leave it on top

Now spread the cards out and three cards will be facing one way and your original bottom card with be facing the other!

(b) *Nearly Perfect Mind Reading?* The Magician gives the Participant ten cards from A to 10, in order. The Participant then performs several Hummer Shuffles, thoroughly messing up the cards. The Magician is blindfolded. Then, the Participant starts reading off the cards in order, from the top of the disordered pile, telling the Magician what card it is. The Magician is able to guess whether the card is face up or face down, with nearly flawless accuracy (much better than 5 correct—the expected number due to random guessing)!

(c) *Royal Flush Hummer*. The Magician takes about half a deck and shows the cards in it to the Participant, who is invited to shuffle them. The magician then apparently messes the cards up further in a random way with respect to orientation (face-up vs. face-down). Then the Magician invites the Participant to continue messing up the cards with some Hummer-type shuffles. Then the Magician deals the cards into two piles, puts them together, and spreads them out. Exactly 5 cards are face-down. They miraculously form a royal flush!

5 *Random Numbers*. The Magician asks the Participant to choose a random number n between 1 and 20, and share this number with the audience without letting the Magician know. The Participant then removes the top n cards from the deck.

Next, the Magician deals 20 cards from the top of the diminished deck (which is missing n cards), and he asks the audience to notice the nth card dealt (without giving it away with body language!).

Next, an audience member is asked to estimate half the size of the now very diminished deck (it is missing $20+n$ cards). We call this number h. The Magician then deals h cards from the top, face-down. Then he places the stack of 20 cards on top of this, and puts the rest of the diminished deck on top of that (so the n cards removed at the start are still missing).

Finally, the Magician deals cards off the top, but at some miraculous point, stops, and it is the one that the audience noted!

The Mysteries, Revealed!

1 *Fingers That Can See*. M watches and keeps track of the total number of face up cards. Call this number u. Then while blindfolded, M merely collects any u cards into a pile (making sure to keep their original orientation) and then flips this entire pile upside-down. Then this pile and the remaining cards have the same number of face-up cards. The reason: suppose that, among the u cards collected, that f of them are face up. Then $u-f$ are face down. However, in the pile of non-collected cards, $u-f$ must be face up (since the total number of face up cards is u). So flipping the chosen cards does what we want!

2 *Zvonkin's Magic Table*. The table is a repeating grid of 5×5 numbers arranged so that each row and each column sums to 20. Such grids are easy to make—try it yourself!—and now the trick is obvious: just look at the number adjacent to the covered area, and subtract this from 20.

3 *The Kruskal Count*. This trick works for the same reason that putting a hotel on Park Place is almost always a winning Monopoly strategy: eventually, someone will land at Park Place!

Pick the very first card (or digit) and plot out the evolution of this pick. Imagine, say, that each card (digit) that gets visited is colored green. Now consider a different starting point. This will engender a new sequence of visited locations. But observe that as soon as we reach a green location, we are locked into all the rest of the green locations.

So now, think of the green locations as "mines" or "Monopoly hotels that belong to our opponent." We start at some random point, and then our course is preordained (by the actual values of the cards or digits) but is also, in some sense, random. With digits, each step will be have length from 1 to 9, with each choice approximately equal (1 is more likely, since landing on 0 leads to a step size of 1). With cards, step sizes of 4 and 5 are somewhat more likely than the others, but otherwise, it's a random choice between 1 and 9.

In other words, each random sequence of digits (or shuffled deck of cards) plus a starting point yields a random sequence of step lengths, with approximately equal probabilities for each step length.

How do you avoid green locations? At each step, look for the nearest green location, and make sure not to step that distance. Just like Monopoly: if you are 8 steps from Park Place, you toss your dice, hoping not to get an 8. Since there are 9 possible step lengths, and only one bad one, at each turn, you have an $8/9$ probability of missing the next green location. Consequently, if you do this 15 times, the probability of missing *all* of the green locations is $(8/9)^{15}$, which is about 17%. Hence there is an 83% probability that you will hit a green spot and then get locked into the sequence that began with the very first location.

So that's how the Magician does the trick, by starting from the first spot and knowing that, with high probability, the Participant and the Magician will end up in the same place.

4 *Hummer Shuffle Tricks*. Consider a pile of cards, where some possibly are face up. Each card has a position (from #1, the top card, down to the last card), a value (where $A=1$ and J,Q,K respectively equal 11, 12, 13), and an orientation (either face-up or face-down). All of these tricks depend on using an *even* number of cards and use one or both of the following lemmas.

> **Lemma 1**: Start with a pile of $2n$ cards, *all face-down*. After any number of Hummer Shuffles is performed, the number of odd-position cards that are face-up will equal the number of even-position cards that are face-up.

> **Lemma 2**: Start with a pile of $2n$ cards, *all face-down*, and *arranged in numerical order* (for example, $5, 6, 7, 8, 9, 10, J, Q$). then do any number of Hummer Shuffles. For each card, the sum of its position, value, and orientation (where we assign 1 to "face-up" and 0 to "face-down") will have the same parity.

For example, suppose the cards start with $5, 6, 7, 8$ from top to bottom, all face-down, and we turn over the first two and cut by taking the top card and putting on the bottom and then turn over the top two and cut by taking the top two cards and putting them on the bottom. Then we get, in order (using a bar to indicate "face-up"), from the starting position:

$$5, 6, 7, 8 \to \bar{6}, \bar{5}, 7, 8 \to \bar{5}, 7, 8, \bar{6} \to \bar{7}, 5, 8, \bar{6} \to 8, \bar{6}, \bar{7}, 5.$$

Now let's compute the sum of position plus value plus orientation for each card. The first card's sum is $1 + 8 + 0 = 9$. Card #2's sum is $2 + 6 + 1 = 9$. Card #3's is $3 + 7 + 1 = 11$, and the final sum is $4 + 5 + 0 = 9$. All of these are odd.

I leave it to the reader to prove these lemmas, but this should not be difficult. The harder part is thinking of the lemmas in the first place! We also leave it to the reader to use these lemmas (or other similar ideas) to explain (a).

Lemma 2 is used for (b), the Nearly Perfect Mind Reader trick. The Magician merely guesses the first answer, but of course the Participant will tell the Magician if he or she is correct or not. This establishes the parity of the sum, and the rest is (fairly) easy, but requires paying attention.

For (c), the Magician makes sure that there are an even number of cards in the pile, and that a royal flush is included among them. Then M cleverly arranges the orientation of the cards by examining successive pairs and flipping over the odd-positioned card ONLY if it belongs to the royal flush, and flipping over the even-positioned card ONLY if it doesn't belong to the royal flush. I am right handed, so I start looking at the cards from the right, so I use the mnemonic aid "**R**oyal flush cards get flipped if they are the **R**rightmost one in the pair."

At this point, some cards are face-up and some are face-down, but the following regularity has been imposed:

> *The odd-positioned royal cards have the same orientation as the even-positioned ordinary cards. Likewise, the even-positioned royal cards have the same orientation as the odd-positioned ordinary cards.*

Notice (VERIFY!) that Hummer Shuffling will not change this situation! So after a bunch of Hummer Shuffles (even ones where you flip over the top 4 cards, or any even number of cards), the Magician finally deals the cards out into two piles, alternating cards. M observes which pile has face-up royal cards, and takes this pile and surreptitiously turns it over and places it on the other pile. Now the only cards that are face down will be the royal cards!

5 *Random Numbers*. The crux idea behind this trick is that $n + (-n) = 0$. Keep it simple for a moment, and suppose that $h = 0$. Then P takes n cards off the top of the deck, and M draws out 20 from the $(52 - n)$-card deck, with the audience noting the nth one. Since $h = 0$, M just puts the $(32 - n)$-card deck on top of the 20-card deck. However, the audience's card is the nth from the top of the 20-card deck. Adding, we get $32 - n + n = 32$; thus M merely counts down to the 32nd card and this will be the target.

In the more general case, there will be h cards at the bottom, 20 cards in the middle (with the target card at the nth position from the top) and $32 - n - h$ cards on top. So now M counts to the $32 - h$th card. Easy!