Tulsa Math Teacher's Circle
February 6, 2014
Grid Luck[1]
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1. (warmup) Tetris pieces are made of all the different ways you can connect 4 squares together. (Rotating doesn't make a difference, but two pieces which are non-identical mirror images are considered different.) Can you remember/figure out all the Tetris pieces?

2. How many squares are there in a $m \times n$ rectangle?

3. How many rectangles are there in a 2x3 rectangle? 5x8 rectangle? How many rectangles in an $m$ by $n$ rectangle?

4. Draw a rectangle and trace the path of a billiard ball that begins in the lower left corner and initially travels upward at a 45-degree angle. Assuming that the ball bounces off the walls at perfect 45-degree angles, which corner does the ball reach first? What fraction of all the unit squares within your rectangle does the ball pass through on its way? Start your experiment with a rectangle having width 3 and height 5, then choose other dimensions. What will happen if the width is $m$ and the height $n$ units for some positive integers $m$ and $n$?

5. Is it possible to dissect a square into 2 squares? 4 squares? 6 squares? 7 squares? Can you decide for which positive integers n it is possible to dissect a given square into n smaller squares?

[1] Adapted from *Grid Luck* with thanks to by Tatiana Shubin, San Jose State University http://www.mathteacherscircle.org/assets/session-materials/TShubinGridLuck.pdf

Notes:

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**Problem 2.** Let's assume that the vertices of the squares are points in the grid and, for now, that their sides are are aligned with the horizontal an vertical grid lines. Can we assume that, without loss of generality that $m \leq n$? If so, then the largest square is $m \times m$ on a side and the smallest is $1 \times 1$.

Is easy to see that there are $m \times n$ squares of side 1 in the rectangle ($m$ verical positions and $n$ horizontal positions). For squares of side $m$ there is only one vertical position and $n - m + 1$ horizontal positions, so, there are $1 \times (n - m + 1)$ such squares in the rectangle. (Try out the formula for a rectangle of size $5 \times 8$.)

What about squares of side $k$ where $1 < k < m$? Now many vertical and horizontal positions? For an example let's consider squares of side 4 in a $5 \times 8$ rectangles. It appears that there are 2 vertical positions and 5 horizontal positions for a count of $2 \times 10 = 20$ such squares.

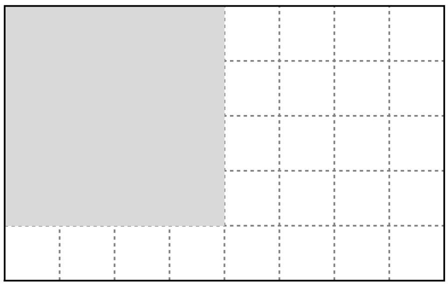

Figure 1: 4x4 Square in 5x8 Rectangle

Now generalize this for rectangles of size $m \times n$ and squares of side $k$, $1 \leq k \leq m$. There are $m-k+1$ vertical positions and $n-k+1$ horizontal positions, so, the number of squares of side $k$ in an $m \times n$ rectangle is

$$(m - k + 1)(n - k + 1) = (m + 1)(n + 1) - k(m + n + 2) + k^2.$$

The total number of squares sizes 1 to $m$ is

$$\begin{aligned}
&\sum_{k=1}^{m}(m - k + 1)(n - k + 1) \qquad (1)\\
&= \sum_{k=1}^{m}\left((m + 1)(n + 1) - k(m + n + 2) + k^2\right)\\
&= m(m + 1)(n + 1) - (m + n + 2)\left(\sum_{k=1}^{m} k\right) + \left(\sum_{k=1}^{m} k^2\right)\\
&= m(m + 1)(n + 1) - (m + n + 2)\left(\frac{m(m + 1)}{2}\right) + \left(\frac{m(m + 1)(2m + 1)}{6}\right)
\end{aligned}$$

Which, after a bit of algebra (a lot maybe?), is

$$\frac{m(m + 1)(3n - m + 1)}{6}. \qquad (2)$$

This is the total number of squares in the $m \times n$ rectangle (with horizontal and vertical sides).

You may recall that the sum of the first $m$ positive integers is

$$1 + 2 + \cdots + m = \sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$

and the sum of the squares of the first $m$ positive integers is

$$1^2 + 2^2 + \cdots + m^2 = \sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}.$$

(If not, add these problems to your mathematics *to do* list.)

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**Problem 2. With a Twist:** Suppose we consider the number of squares in an $m \times n$ rectangle as before, but allow any squares with vertices on the grid. For example, consider such squares inscribed in a $5 \times 5$ square.

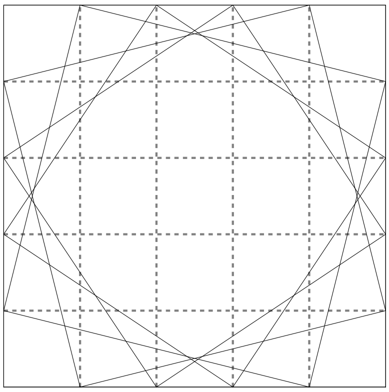

Figure 2: Squares inscribed in a $5 \times 5$ square.

That is, for every $5 \times 5$ square aligned with the horizontal and vertical, there are four more inscribed in it for total of $4 + 1 = 5$ squares. Can you generalized this example to a $k \times k$ square? If so, then how would you generalize equation (1) to account for all squares in an $m \times n$ rectangle regardless of orientation? Then, how would you derive a formula annalogus to equation (2)? (You may need to add the closed form of the sum $1^3 + 2^3 \cdots m^3$ to your list.)

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**Problem 3.** How many rectangles are there in an $m \times n$ rectangle. For now assume that the vertices of the rectangles are on the grid and the sides are verticle and horizontal. For an example, consider the number of $1 \times 3$ rectangles contained in a $3 \times 5$ rectangle.

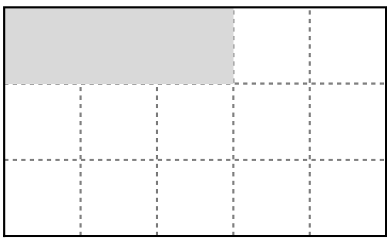

Figure 3: $1 \times 3$ Rectangles inscribed in a $3 \times 5$ rectangle.

There are $3 = 3 - 1 + 1$ vertical positions and $3 = 5 - 3 + 1$ horizontal positions yielding 9 rectangles. To generalize this example, consider a rectangle of fixed size $m \times n$ and any rectangles of smaller dimension, say, $k \times l$, $1 \leq k \leq m$ and $1 \leq l \leq n$. Is it true that there are

$$(m - k + 1)(n - l + 1) \tag{3}$$

such rectangles? If so, then the answer to this problem can be found by suming over all values of $k$ and $l$

$$\sum_{k=1}^{m} \sum_{l=1}^{n} (m - k + 1)(n - l + 1). \tag{4}$$

Since the first term of the product depends only upon the summation index $k$ and the second only on the index $l$ the double summation can be written as

$$\left( \sum_{k=1}^{m} (m - k + 1) \right) \left( \sum_{l=1}^{n} (n - l + 1) \right) = \frac{n(n+1)}{2} \frac{m(m+1)}{2} \tag{5}$$

since $\sum_{k=1}^{m} (m - k + 1)$ and $\sum_{l=1}^{n} (n - l + 1)$ are simply the sum of the first $m$ and $n$ positive integers respectively. (This result seems to pop up a lot lately.)

An interesting geometric interpretation of this product can be found in the attached graphic labeled *Multiplication Table*. The number of rectangles in a $4 \times 7$ rectangle can be found by counting the number of unit squares in the part of the graphic displayed in Figure 4 which is a subsection of the larger table.

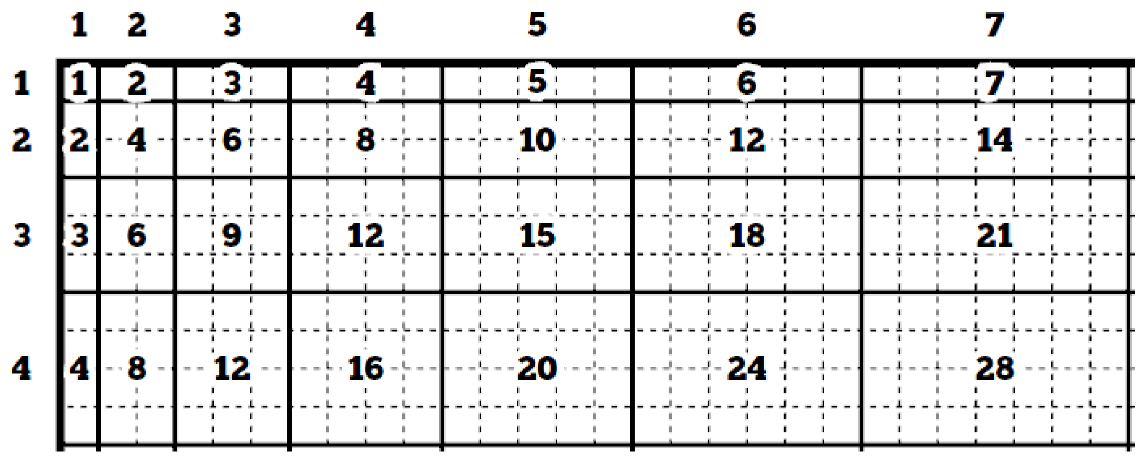


Figure 4: $4 \times 7$ section of the *Multipication Table*

The number of rectangles from equation (5) is

$$\frac{4 \times 5}{2} \frac{7 \times 8}{2} = 280$$

which is also the number of the unit squares in Figure 4 as well as the sum of the products in the *multiplication table*. In this example, the 1 in the upper left corner represents the single $4 \times 7$ rectangle and the the 10 unit squares in the product $2 \times 5$ represent the each of the possible $(4 - 3 + 1) \times (7 - 3 + 1) = 2 \times 5$ rectangles in the $4 \times 7$ rectangle.

**Problem 4.** In addition to the questions posed in this problem consider the following:

**a)** Suppose the width of the table is $3m$ and the height is $3n$. Will the ball end up in the same pocket as a table with width $m$ and height $n$? Hence can we restrict our investigation to those tables for which the the width and height are relatively prime ($\gcd(width, height) = 1$)?

**b)** Trace the path of the ball on the example tables on the next page and note the pocket. Can you pose a conjecture about the tables for which the ball ends up in pocket A? Pocket B? Pocket C? Is it possible that the ball could end up in the corner pocket at the starting point?

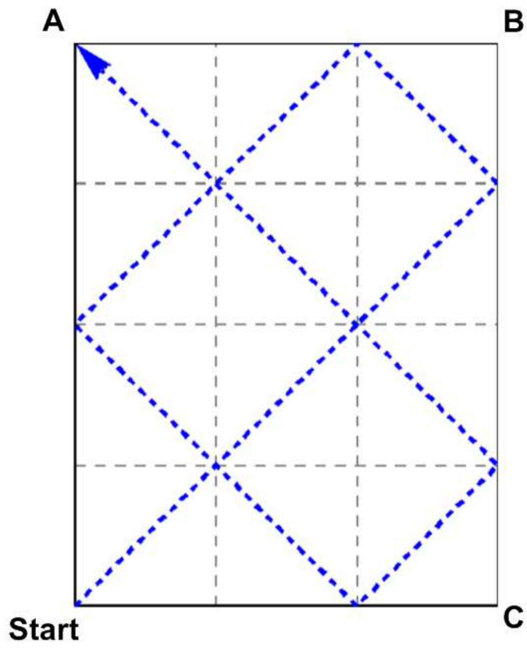


Figure 5: $4 \times 3$ Billiard Table

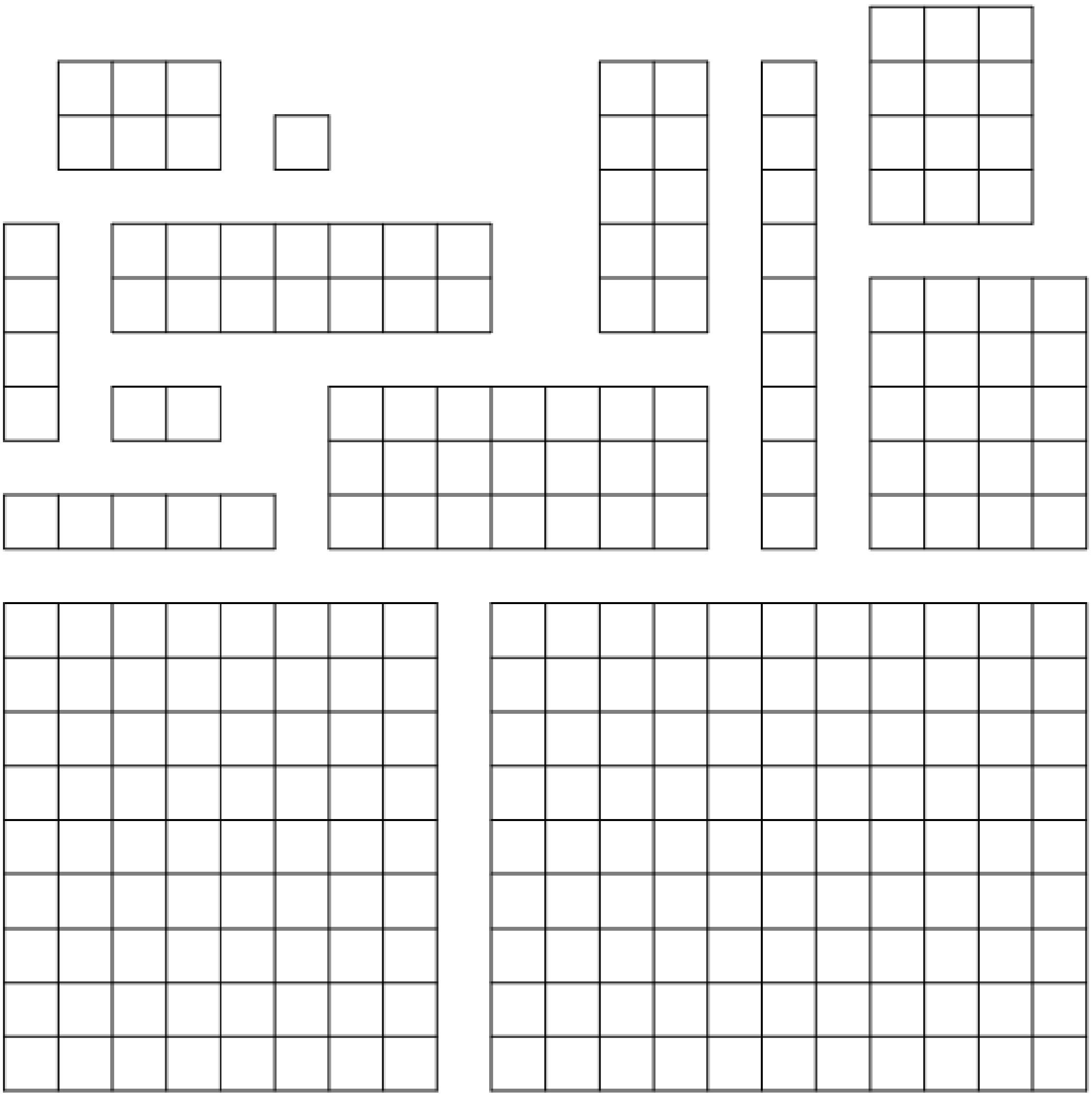

Figure 6: Examples of Billiard Tables

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**Problem 5.** Determine whether or not it is possible to dissect a square into $n$ squares (possibly different sizes) for $n = 2, 3, \ldots, 10$. If it is known that a square can be divided in to $n > 10$ squares, what can you say about $n + 3$? About $n + 5$? Can you now answer for all positive integers?

# Multiplication Table
## (Drawing is to scale...)

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 9 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 10 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

the griddle
handcrafted logical fun

Worksheet design by David Millar, http://thegriddle.net/educators
Table concept patented by SilabuS, http://www.silabus.hr