Inspiring Mathematics:
Lessons from the Navajo Nation Math Circles

Dave Auckly
Robert Klein
Amanda Serenevy
Tatiana Shubin

The Triangle tilings

We are used to seeing areas tiled by rectangles. It is less common to see areas tiled by triangles, but there are many interesting mathematical problems related to such tilings. In this activity, we will explore several and make some awesome models. Pick three positive integers (p, q, r) and make a triangle with angles equal to the fractions $1/p$, $1/q$ and $1/r$ of a straight line.

Put one copy on the table, and then place new copies according to the rule: each new triangle must be a reflection of one of the triangles already placed through an edge of that triangle. See Figure 1 below.

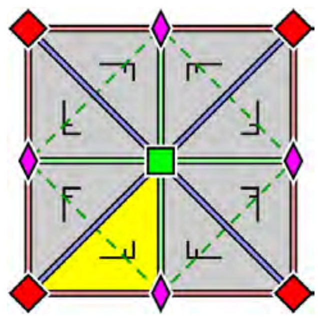

Figure 1: Reflecting a triangle CITE WIKI IMAGE

(1) What is the triple (p, q, r) for Figure 1?

(2) Make the fancy sphere model. What is the triple (p, q, r) for this model?

(3) Look at the following tiling with the $(2, 3, 6)$ pattern. Notice that the blue and red edges have arrows on them. Also notice the green dot.

   (a) How would you rotate the diagram (where would the center of rotation be and through what angle would you rotate?) to move the green dot along the red edge to the next junction? (Call this the R rotation.)

   (b) How would you rotate the diagram (where would the center of rotation be and through what angle would you rotate?) to move the green dot along the blue edge to the next junction? (Call this the B rotation.)

   When we compose operations (first do one and then the other) we work from right to left. For example if T is the operation given by $T(x) = x+1$ and S is the operation given by $S(x) = -1/x$ then $S \circ T(x) = S(T(x)) = S(x + 1) = -1/(x + 1)$. Similarly, RBB represents the result of doing

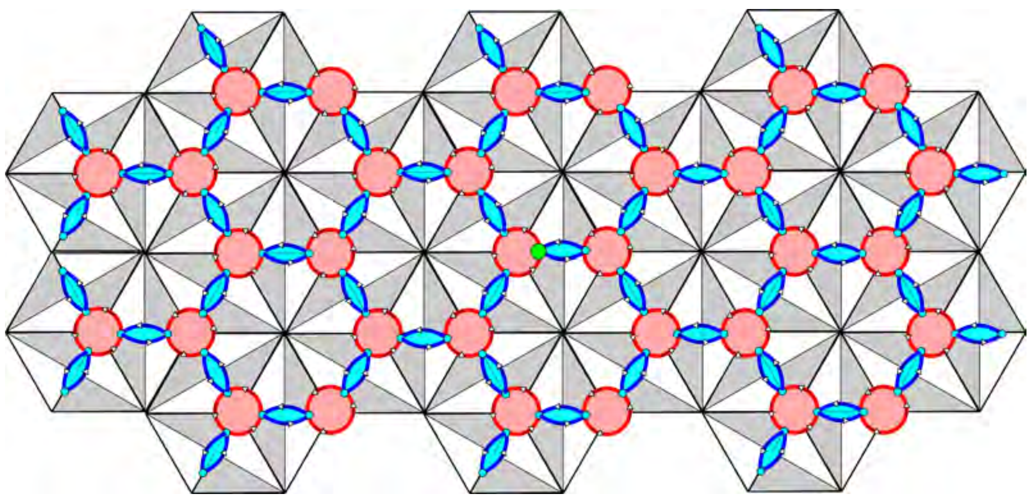

Figure 2: A $(2,3,6)$ tiling

the blue rotation, then repeating the blue rotation and then doing the red rotation. We can also see this as a path along the edges. The RBB path would first travel along the **red** edge, then travel along a blue edge and finally along the next blue edge.

(4) Draw the RBB path and explain how to construct a path using a sequence of rotations. There should be a one-to-one and onto correspondence between paths and sequences of rotations. (Thus we can either think about sequences of rotations or paths along the edges.)
The (p,q,r) triangle group is the collection of all sequences of reflections in the sides of a (p,q,r) triangle. This group is denoted by $\widehat{\Delta}(p,q,r)$. The subgroup of orientation preserving motions is denoted by $\Delta(p,q,r)$. It is generated by $2\pi/p$, $2\pi/q$, and $2\pi/r$ (radian) rotations about the respecitve corners (and thier inverses). In fact it is generated by any two of these rotations.

(5) Explaning how to get the $1/6$th rotation out of a sequence of red and blue rotations.
In a group a distinguished set of elements is called a *generating set* if every element of the group may be expressed as a (finite) sequence of generators and their inverses. From here forward we will leave out the expression "and their inverses."

(6) Show that R moves any corner of any trinagle in the tiling to some other corner. (The same is true for B.)

(7) Let P and Q are rotations about points p and q respectively. Show that PQP^{-1} is a rotation about the point $P(q)$. (Conclude that the only element of the triangle group $\Delta(2,3,6)$ that fixes the green point is the trivial element.) The corresponding paths are the ones that start and end at the green point.

A relation is a (finite) sequence of generators representing the trivial element. For example $RB^{-1}RR^{-1}BR^{-1}$. In fact any sequence of generators followed by the sequence of inverse generators in the reverse order will be a relator. We summarize this by $gg^{-1} = 1$. In the $\Delta(2,3,6)$ triangle group we also have the relations $R^3 := RRR = 1$ and $B^2 = 1$.

Given any relation $r = 1$, adding a sequence and its inverse will lead to a new relation, $grg^{-1} = 1$. Furthermore given any two relations $r_1 = 1$ and $r_2 = 1$ the expression $r_1 r_2 = 1$ is a relation and the expression r_1^{-1} is a relation.

We say a sequence of relations generate all relations if any relation may be obtained from elements in the sequence by repeated application of the rules that generate new relations from old.

(8) Draw a lasso along the edges of of Figure 2. Write out the corresponding relation and explain how to generate it out of the relations $R^3 = 1$, $B^2 = 1$ and $(BR)^6 = 1$.

(9) Explain why all the relations in the group $\Delta(p,q,r)$ are generated by the relations $R^3 = 1$, $B^2 = 1$ and $(BR)^6 = 1$.

(10) Look at the following image of the $(2,3,\infty)$ trinagle tiling. Let S be the 180° rotation about the center of the top left "eyebrow." Let P be the 1/3rd clockwise rotation about the center covered by the center top red oval. Find a generating set of relations in the generators S and P.